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Non-symmetric, Metric, Cometric Association Schemes Are Self-Dual

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It is shown here that non-symmetric, metric, cometric association schemes are self-dual, and that the parameters and eigenvalues can be recursively calculated in terms of roughly $\frac{1}{2}d$ integer parameters. This is in marked contrast to the symmetric case in which such schemes need not be self-dual, but the parameters and eigenvalues are functions of 9 parameters independent of the diameter d.

The work on closed forms for the parameters and eigenvalues of symmetric, metric, cometric association schemes was done in [3–5]. For a thorough background of the subject refer to Bannai and Ito [1], which contains the material in [3, 4]. Reference [6], which gives closed forms for the parameters in terms of the eigenvalues and dual eigenvalues in the non-symmetric case, contains all the background and notation needed to understand this paper. For a non-trivial example of such schemes see Liebler and Mena [7]. See Damerell [2] for general background.

It is shown here that non-symmetric, metric, cometric association schemes are self-dual. It is further shown that roughly $\frac{1}{2}d$ parameters suffice to determine the rest, and that the first eigenvalue satisfies a quadratic equation over the rationals (unless the scheme is a directed cycle.)

As in [6], let $P_i(x)$, $0 \leqslant i \leqslant d$ be a graded set of polynomials defined recursively by $P_0(x) = 1$, $P_1(x) = x$, and

$$xP_i(x) = \sum_{j=1}^{i+1} p_{1ij} P_j(x), \; p_{1i(i+1)} > 0, \qquad 1 \leqslant i < d.$$

For fixed $x_0, \ldots, x_{d-1}$ let $p_j(x) = \prod_{k=0}^{j-1} (x - x_k)$, $0 \leqslant j \leqslant d$, and write $P_i(x) = \sum_{j=0}^{i} L_{i,j} p_j(x)$ relative to this new basis. These equations define a

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$(d+1)\times(d+1)$ lower triangular matrix $L(x_0, ..., x_{d-1})$ with (i, j)th entry $L_{i,j}$, which was shown in [6] to satisfy the matrix equation

$$L^{-1}(\theta_1, ..., \theta_d)\,\mathrm{diag}(\theta_0^*, ..., \theta_d^*)=\mathrm{diag}(\theta_0^*, ..., \theta_d^*)L^{-1}(\theta_0, ..., \theta_{d-1}). \tag{1}$$

This equation was not exploited in the previous paper, but will be used here to show the self-duality of the scheme.

Let Δ^j be the jth iteration of the generalized difference operator Δ defined recursively by $\Delta^0 f(x_0)=f(x_0)$ and

$$\Delta^j f(x_0, ..., x_j)=\frac{\Delta^{j-1}f(x_0, ..., x_{j-2}, x_j)-\Delta^{j-1}f(x_0, ..., x_{j-1})}{x_j-x_{j-1}}.$$

Then since $P_k(x)=xR_{k-1}(x)$ for some polynomial $R_{k-1}(x)$ of degree $k-1$ when $k>0$,

$$L_{k,j}=\Delta^j P_k(x_0, ..., x_j)=\begin{cases} P_k(x_0) & \text{if } j=0\\ x_j\Delta^j R_{k-1}(x_0, ..., x_j)+\Delta^{j-1}R_{k-1}(x_0, ..., x_{j-1}) \\ \quad \text{if } j>0. \end{cases} \tag{2}$$

LEMMA. $(L^{-1})_{t,0}=\prod_{l=0}^{t-1}(-x_l)$, $0\leqslant t\leqslant d$.

Proof (by induction). $L_{0,0}=1$, so $(L^{-1})_{0,0}=1$ because L is lower triangular. Suppose inductively that $(L^{-1})_{k,0}=\prod_{l=0}^{k-1}(-x_l)$ for all $k<t$. Since $LL^{-1}=I$ has $(t, 0)$th entry 0 for $t>0$,

$$\begin{aligned} 0=\sum_{k=0}^{t} L_{t,k}(L^{-1})_{k,0}&=(L^{-1})_{t,0}\Delta^t P_t(x_0, ..., x_t)\\ &\quad+\sum_{k=0}^{t-1}\prod_{l=0}^{k-1}(-x_l)\Delta^k P_t(x_0, ..., x_k)=(L^{-1})_{t,0}\Delta^t P_t(x_0, ..., x_t)\\ &\quad+\prod_{l=0}^{t-2}(-x_l)x_{t-1}\Delta^{t-1}R_{t-1}(x_0, ..., x_{t-1}). \end{aligned}$$

Since $\Delta^t P_t(x_0, ..., x_t)=\Delta^{t-1}R_{t-1}(x_0, ..., x_{t-1})$ is the leading coefficient of $P_t(x)$ (which is non-zero), the lemmas follows.

THEOREM. *Any non-symmetric, metric, cometric association scheme with* $g=d+1$ *is self-dual.*

Proof. Then from the $(t, 0)$th entry of the matrix equation (1)

$$\theta_0^*\prod_{l=0}^{t-1}(-\theta_{l+1})=\theta_t^*\prod_{l=0}^{t-1}(-\theta_l). \tag{3}$$

Since $\theta_l \neq 0$ for all l,

$$\theta_t^*/\theta_0^* = \theta_t/\theta_0, \qquad 0 \leqslant t \leqslant d. \tag{4}$$

Then from the closed form for the parameters found in [6] (with $b_j := p_{1(j+1)'j'}$),

$$\frac{b_j}{\theta_0} = \frac{b_j^*}{\theta_0^*},$$

so

$$\frac{m_j}{\theta_0^*} = \frac{v_j}{\theta_0}.$$

And since $\sum_{j=0}^d m_j = \sum_{j=0}^d v_j = n$, $\theta_0^* = \theta_0$, so the scheme is self-dual.

Then $a_j := p_{1jj} = (\theta_j - \theta_0)(\theta_j^2 - \theta_{j-1}\theta_{j+1})/(\theta_j - \theta_{j-1})(\theta_j - \theta_{j+1})$ gives

$$\theta_{j+1} = \theta_j \frac{\theta_j(\theta_j - \theta_0) - a_j(\theta_j - \theta_{j-1})}{\theta_{j-1}(\theta_j - \theta_0) - a_j(\theta_j - \theta_{j-1})}$$

so the eigenvalues (and hence all of the parameters) can be determined recursively given θ_0, θ_1, $a_1, a_2, ..., a_d$. Since $a_{j'} = a_j$ only roughly half of these are needed.

From the (2, 1), (3, 1), and (3, 2) entries of the matrix equation (1)

$$\begin{aligned} b_1(\theta_1 + b_1 - a_2)(\theta_1 + b_1 - a_2 - b_2) \\ = b_2(\theta_1 + b_1)(\theta_1 - \theta_0 + b_1). \end{aligned}$$

If $b_1 \neq b_2$ this gives θ_1 in terms of θ_0, b_1, b_2, and a_2. If $b_1 = b_2$ and θ_1 is not rational, then $a_1 = a_2 = 0$, $b_0 = b_1 = b_2$.

T. Ito has noted that these results can be derived also by suitably generalizing arguments in [4] to non-symmetric schemes.

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